Chapter six:

6. Riemann Integral (تكامل ريمان)

6.1 Definitions and Characterizations (تعاريف واوصاف)

Definition 6.1:

1- A partition of the interval $[a, b]$ is a set

$$\rho = \{a = x_0, x_1, x_2, \dots, x_n = b\} \text{ such that } x_0 < x_1 < x_2 < \cdots < x_n.$$

2- $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ are called segments of ρ.

3- $\Delta x_i = |x_{i-1} - x_i|$, $\forall i = 1,2, \dots, n$ is called the length of the segment $[x_{i-1}, x_i]$.

4- $\Delta\rho = \max \{\Delta x_i \mid i = 1,2, \dots, n\}$ is called the norm of ρ.

5- If $\rho^* = \{x_0, x_1, x_2, y, x_3, z, x_4, \dots, x_n\} \Longrightarrow \rho \subset \rho^*$.

Definition 6.2: A partition ρ^* is called a refinement of ρ if $\rho \subset \rho^*$ and $\Delta\rho^* \leq \Delta\rho$.

Now, let f be a bounded function on $[a, b]$ and let

$$m = \inf\{f(x) \mid x \in [a, b]\}$$

$$M = \sup\{f(x) \mid x \in [a, b]\}$$

Since f is bounded function on $[a, b]$, then f is bounded on each $[x_{i-1}, x_i]$ and let

$$m_i = \inf \{f(x) \mid x \in [x_{i-1}, x_i]\}$$

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

$$\Longrightarrow m \leq m_i \leq M_i \leq M \quad \forall i = 1,2, \dots, n$$

Let $\underline{R}(f,\rho) = \sum_{i=1}^{n} m_i \Delta x_i$ = Lower Riemann sum of f on $[a,b]$ with a partition ρ.

Let $\overline{R}(f,\rho) = \sum_{i=1}^{n} M_i \Delta x_i$ = upper Riemann sum of f on $[a,b]$ with partition ρ.

Remarks 6.3: 1- $\underline{R}(f,\rho) \leq \overline{R}(f,\rho)$, (since $m_i \leq M_i, \ \forall i$).

2- $m(b-a) \leq \underline{R}(f,\rho) \leq \overline{R}(f,\rho) \leq M(b-a)$.

Remarks 6.4:

1- $\because \underline{R}(f,\rho) \leq M(b-a) \Longrightarrow \underline{R}(f,\rho)$ is bounded above by $M(b-a) \Longrightarrow \underline{R}(f,\rho)$ has *sup*. *i.e.*

$$sup. \quad \{\underline{R}(f,\rho) | \rho \text{ is a partition on } [a,b]\} = R\int_{-} f = \int_{\underline{a}}^{b} f =$$

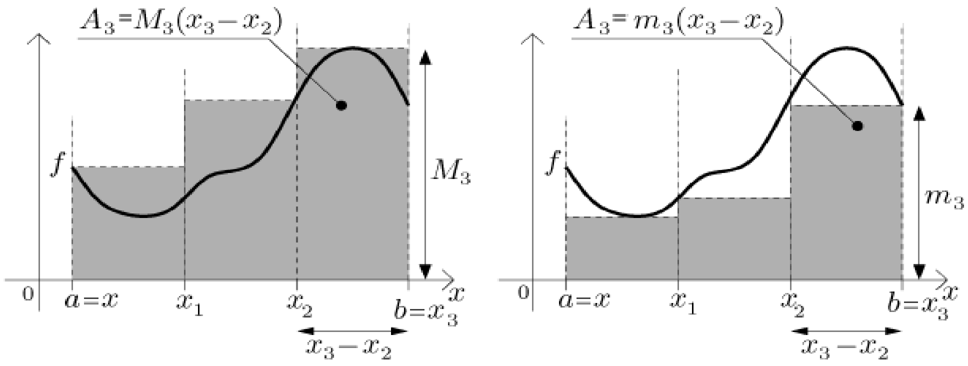
Lower Riemann integral of f on $[a,b]$.

it is clear that $\underline{R}(f,\rho) \leq \int_{\underline{a}}^{b} f$.

2- Since $m(b-a) \leq \overline{R}(f,\rho) \Longrightarrow \overline{R}(f,\rho)$ is bounded below by $m(b-a) \Longrightarrow \overline{R}(f,\rho)$ has *inf* *i.e.*

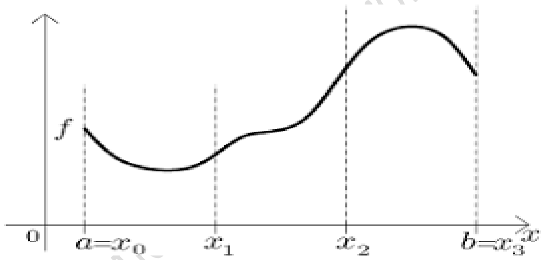
inf $\{\overline{R}(f,\rho) | \rho \text{ is a partition on } [a,b]\} = R\int^{-} f = \int_{a}^{\overline{b}} f$ =upper Riemann integral of f on $[a,b]$.

It is clear that $\int_{a}^{\overline{b}} f \leq \overline{R}(f,\rho)$.



3- $m(b-a) \leq \underline{R}(f,\rho) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \overline{R}(f,\rho) \leq M(b-a)$.

Definition 6.5: Let $f:[a,b] \to R$ be a bounded function on $[a,b]$. If $\underline{\int_a^b} f = \overline{\int_a^b} f$,then f is said to be Riemann integrable (or R-integrable) on $[a,b]$ and denoted by $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$.



Theorem 6.6: If f is a bounded function on $[a,b]$ and ρ^* is a refinement of ρ, then $\underline{R}(f,\rho) \leq \underline{R}(f,\rho^*)$ and $\overline{R}(f,\rho^*) \leq \overline{R}(f,\rho)$.

Proof: Let $\rho = \{a = x_0, x_1, x_2, \dots, x_{u-1}, x_u, \dots, x_{n-1}x_n = b\}$ be a partition on $[a,b]$.

Let $\rho^* = \{x_0, x_1, x_2, \dots, x_{u-1}, \overline{x}, x_u, \dots, x_{n-1}x_n\}$ be a refinement of ρ where $x_{u-1} < \overline{x} < x_u$.

To prove that

$\underline{R}(f,\rho) \leq \underline{R}(f,\rho^*)$, then

$\underline{R}(f,\rho) = \sum_{i=1}^{n} m_i \Delta x_i$ where m_i is $inf\, f$ on $[x_{i-1}, x_i]$.

Let $m' = inf\, f$ on $[x_{i-1}, x_i]$ and $m" = inf\, f$ on $[\overline{x}, x_u]$,

And $m_u = inf\, f$ on $[x_{u-1}, x_u] \Longrightarrow m_u \le m', m_u \le m"$

$\underline{R}(f, \rho^*) = \sum_{i=1}^{u-1} m_i \Delta x_i + (\overline{x} - x_{u-1})\, m' + (x_u - \overline{x}) m" + \sum_{i=u+1}^{n} m_i \Delta x_i$

$\underline{R}(f, \rho^*) = (\sum_{i=1}^{u-1} m_i \Delta x_i + m_u(x_u - x_{u-1}) + \sum_{i=u+1}^{n} m_i \Delta x_i)(\overline{x} - x_{u-1}) m' + (x_u - \overline{x}) m" - m_u(x_u - x_{u-1})$

$= \sum_{i=1}^{n} m_i \Delta x_i + (m' - m_u)(\overline{x} - x_{u-1}) + (m" - m_u)(x_u - \overline{x})$

$\therefore \underline{R}(f, \rho^*) = \underline{R}(f, \rho) + (m' - m_u)(\overline{x} - x_{u-1}) + (m" - m_u)(x_u - \overline{x})$

$\therefore \underline{R}(f, \rho^*) - \underline{R}(f, \rho) = (m' - m_u)(\overline{x} - x_{u-1}) + (m" - m_u)(x_u - \overline{x})$

$\Longrightarrow \underline{R}(f, \rho^*) - \underline{R}(f, \rho) \ge 0 \Longrightarrow \underline{R}(f, \rho) \le \underline{R}(f, \rho^*)$

To prove that $\overline{R}(f, \rho^*) \le \overline{R}(f, \rho)$

Since $\overline{R}(f, \rho) = \sum_{i=1}^{n} M_i \Delta x_i$ *where* M_i *is* $sup\, f$ on $[x_{i-1}, x_i]$.

Let $M' = sup\, f$ on $[x_{u-1}, \overline{x}]$ *and* $M" = sup\, f$ on $[\overline{x}, x_u]$

and $M_u = sup\, f$ *on* $[x_{u-1}, x_u] \Longrightarrow M' \le M_u \le, M'' \le M_u$

$\overline{R}(f, \rho^*) = \sum_{i=1}^{u-1} M_i \Delta x_i + (\overline{x} - x_{u-1}) M' + (x_u - \overline{x}) M'' + \sum_{i=u+1}^{n} M_i \Delta x_i$

$\overline{R}(f, \rho^*) = (\sum_{i=1}^{u-1} M_i \Delta x_i + M_u(x_u - x_{u-1}) + \sum_{i=u+1}^{n} M_i \Delta x_i)(\overline{x} - x_{u-1}) M' + (x_u - \overline{x}) M'' + M_u(x_u - x_{u-1})$

$= \sum_{i=1}^{n} M_i \Delta x_i + (M' - M_u)(\overline{x} - x_{u-1}) + (M'' - M_u)(x_u - \overline{x})$

$\therefore \overline{R}(f, \rho^*) = \overline{R}(f, \rho) + (M' - M_u)(\overline{x} - x_{u-1}) + (M'' - M_u)(x_u - \overline{x})$

$\therefore \overline{R}(f, \rho^*) - \overline{R}(f, \rho) = (M' - M_u)(\overline{x} - x_{u-1}) + (M'' - M_u)(x_u - \overline{x})$

$\Rightarrow \overline{R}(f,\rho^*) - \overline{R}(f,\rho) \le 0 \Rightarrow \overline{R}(f,\rho^*) \le \overline{R}(f,\rho)$

Corollary 6.7: If f is a bounded function on $[a,b]$, then $\underline{R}(f,\rho_1) \le \overline{R}(f,\rho_2)$ for any partitions ρ_1 and ρ_2 of $[a,b]$.

Proof: Let $\rho = \rho_1 \cup \rho_2 \Rightarrow \rho$ is a refinement of $\rho_1$and ρ_2

$\underline{R}(f,\rho_1) \le \underline{R}(f,\rho)$ (by theorem(1))……….(1)

$\underline{R}(f,\rho) \le \overline{R}(f,\rho)$ (by definition)………...(2)

$\overline{R}(f,\rho) \le \overline{R}(f,\rho_2)$ (by theorem)…………(3)

From (1),(2) and (3) we get

$\underline{R}(f,\rho_1) \le \overline{R}(f,\rho_2)$.

Remarks 6.8:

1-$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

2-$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

3-$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n(n+1)}{2}$

6.2 Examples

1- Let $f:[a,b] \to R$ be constant function such that $f(x) = k \ \forall x \in [a,b]$, $k \in R$, then f is R-integrable on $[a,b]$.

Solution 1:

Let $\rho = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition on $[a,b]$.

$\underline{R}(f,\rho) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} k\Delta x_i = k\sum_{i=1}^{n} \Delta x_i = k(b-a)$

$\therefore \underline{\int_a^b} f = \sup\{\underline{R}(f,\rho) \mid \rho \text{ is a partition on } [a,b]\}$

$= \sup\{k(b-a), k(b-a), \dots\}$

$= k(b-a),$

And $\overline{R}(f,\rho) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} k\Delta x_i = k\sum_{i=1}^{n} \Delta x_i = k(b-a)$

$\therefore \overline{\int_a^b} f = \inf \quad \{\overline{R}(f,\rho) \mid \rho \text{ is a partition on } [a,b]\}$

$= \inf\{k(b-a), k(b-a), \dots\}$

$= k(b-a).$

$\because \underline{\int_a^b} f = \overline{\int_a^b} f = k(b-a) \Longrightarrow f is\ R -$ integrable on $[a,b]$.

2-Let $f: [a,b] \to R$ be a function such that

$$f(x) = \begin{cases} 0 & x \in Q \text{ in } [a,b] \\ 1 & x \notin Q \text{ in } [a,b] \end{cases}$$

is f R-integrable on $[a,b]$?.

Solution 2:

Let $\rho = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition on $[a,b]$.

$\underline{R}(f,\rho) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0\Delta x_i = 0 + 0 + \cdots + 0 = 0$

$\therefore \underline{\int_a^b} f = \sup\{\underline{R}(f,\rho) \mid \rho \text{ is a partition on } [a,b]\}$

$= \sup\ \{0,0,0,\dots.\} = 0$

And $\overline{R}(f,\rho) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} 1\Delta x_i = \sum_{i=1}^{n} \Delta x_i = (b-a)$

$\therefore \overline{\int_a^b} f = \inf \quad \{\overline{R}(f,\rho) | \rho \text{ is a partition on } [a,b]\}.$

$= \inf\{(b-a),(b-a),\dots\}.$

$= (b-a).$

$\because \underline{\int_a^b} f \neq \overline{\int_a^b} f \Longrightarrow f$ is not $R-$ integrable on $[a,b]$.

3- Let $f:[a,b] \to R$ be a +function such that

$f(x) = \begin{cases} -3 & x \in Q \text{ in } [a,b] \\ 2 & x \notin Q \text{ in } [a,b] \end{cases}$ is f R-integrable on $[a,b]$

Solution 3: (check)