Definition 4.9:

Let (X, d) be a metric space (or topological space), $S \subseteq X$ and $p \in X$.

We say that P is a limit point (or cluster point or accumulation point) of $S \Leftrightarrow \forall r > 0$, $(B_r(p) \setminus \{p\} \cap S \neq \emptyset$.

Remark 4.5:

1- The set of all limit points of S is denoted by$\acute{S}$.

2- $\bar{S} = S \cup \acute{S}$ is called the closure of S.

Example 4.10:

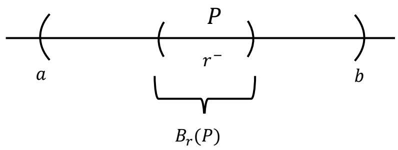
Let (R, d) be the usual metric space.

1- If $S^1 = (a, b) \Rightarrow \acute{S} = [a, b]$ & $\bar{S} = [a, b]$.

Proof:

1- Let $p \in (a, b) \Rightarrow \forall r > 0, (B_r(p) \setminus \{p\}) \cap S \neq \emptyset$

$\Rightarrow p$ is a limit point of S



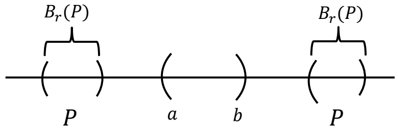
2- If $p = a$ or $p = b$

$\Rightarrow \forall r > 0,\ (B_r(p) \setminus \{p\}) \cap S \neq \emptyset$

$\Rightarrow$ limit point of S

3- If $p \notin (a, b)[p \neq a \forall\, p \neq b]$

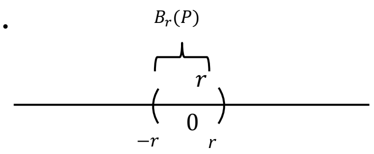
$\Rightarrow \forall r > 0, (B_r(p) \setminus \{p\}) \cap S = 0$



$\Rightarrow p$ is not limit point of S

$\therefore \acute{S} = [a,b] \forall \quad \bar{S} = S \cup \acute{S} = [a,b]$

2- If $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\} \subset R$, then $\acute{S} = \{0\}$.



Solution: Let $x \in R$. If $x = 0$, then

If $B_r(0) = (-r, r)$, then $r > 0 \Rightarrow \exists n \in N$ s.t $\frac{1}{n} < r$

Thus $(B_r(0) \setminus \{0\}) \cap S \neq \emptyset \Rightarrow 0 \in \acute{S}$

If $x \neq 0 \Rightarrow \exists r > 0$ s.t $(B_r(x) \setminus \{x\}) \cap S = \emptyset \Rightarrow x \notin \acute{S}$ (How?)

$\therefore \acute{S} = \{0\}$

3-Let Q be the set of rational numbers and $p \in R$.

$\therefore \forall r > 0, B_r(p) = (p - r, p + r)$
{contains infinite numbers of $Q \& Q^c$}

$\Rightarrow (B_r(p) \setminus \{p\} \cap Q \neq \emptyset \Rightarrow p \in \acute{Q} \Rightarrow \acute{Q} = R$ and $\bar{Q} = R$.

4- If S any finite set in R^n, then $\acute{S} = \emptyset$.

Proof:

Let $S = \{p_1, p_2, \dots p_n\}$ and $q \in R^n$

1- If $q \notin S$

$\Rightarrow \exists r > 0$ s.t $d(q, P_i) > r, i = 1, 2, \cdots, \cdots$

Thus $B_r(q) \cap S = \emptyset \Rightarrow q \notin \acute{S}$

2- If $q \in \acute{S}$

Let $r = \min\{d(p_i, q), i = 1, 2, \cdots, \cdots\}$

$\therefore \left(B_{\frac{r}{2}}(q)\middle|\{q\}\right) \cap S = \emptyset \Rightarrow q \notin \acute{S}$

From 1 and 2 $\Rightarrow \acute{S} = \emptyset$

5- If $H = \{(x, y) \in R^2 | y > 0\}$, then

$\bar{H} = \{(x, y) \in R^2 | y \geq 0\}$.

6- If $S = z$(inteyer numbers) $\subset R$, then $\acute{S} = \emptyset$ (why?).

Proposition 4.3:

Let X be a metric space (or topological) and $S \subseteq X$. Then S is closed $\Leftrightarrow \acute{S} \subseteq S$.

Proof:

$\Rightarrow$ Let S be a closed set $\Rightarrow S^c$ open set

Let $p \notin S \Rightarrow p \in S^c$

$\therefore p$ interior point of S^c

$\Rightarrow \exists r_\circ > 0$ s.t $B_{r_\circ}(p) \subset S^c \Rightarrow (B_r(p)|\{p\}) \cap S = \emptyset$

$\therefore p$ not limit point of S (i.e., $p \notin \acute{S}$)

$\Rightarrow \acute{S} \subseteq S$.

$\Leftarrow$ Let $\acute{S} \subset S$, to prove that S is a closed set (i.e., T.P S^c is an open set).

Let $x \in S^c \Rightarrow x \notin S \Rightarrow x \notin \acute{S}$ (because $\acute{S} \subseteq S$)

$\therefore x$ is not limit point of S.

$\Rightarrow \exists$ open set $\cup_x$ in X s.t

$x \in \cup_x \cap S = \emptyset \Rightarrow \cup_x \subseteq S^c$

It is clear that $S^c = \cup_{x \in S} \cup_x \Rightarrow S^c$ open set.

لاجل ان نفهم طبيعة نقاط التجمع (limit point) في الفضاءات المترية بصورة أوضح نعطي الاتي (المبرهنة التالية توضح الربط بين مفهوم النقطة الحدودية ونقطة التجمع).

Theorem 4.5:

Let X be a metric space, $\emptyset \neq S \subseteq X$ and $p \in X$. (check)

1- If $p \in \acute{S} \Rightarrow p \in S$ or p boundary point of S

2- If p boundary point of $S \Rightarrow p \in S$ or $p \in \acute{S}$

Definition 4.10: (Distance between point and set)

Let X be a metric space and $\emptyset \neq S \subseteq X$, $p \in X$.

Define $A = \{d(p,x) | x \in S\}$

$\because d \geq 0$, then *a* set *A* bounded below by 0.

Thus *A* has great lower bounded (g. l. b) (inf.).

Put $d(p,S) = \inf\{d(p,x) | x \in S\}$

The number $d(p,S)$ is the distance between a point *p* and *a* set *S*.

Remark 4.6:

From Def.4.10, we see that

1- $d(p,S) \geq 0$

2- If $p \in S \Rightarrow d(p,S) = 0$ (العكس غيرصحيح)

3- If $d(p,S) = 0 \nRightarrow p \in S$.

Example 4.11:

1- Let $S = (a,b) \subset R$, then $d(b,S) = 0$ and $d(a,S) = 0$, but $a,b \notin S$.

2- Let $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ in R, then $d(2,S) = 1$ and $d(0,S) = 0$.

Theorem 4.6:

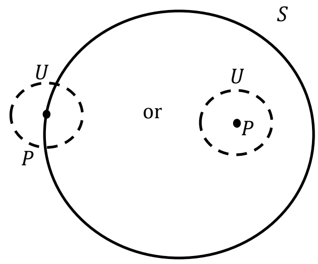
Let X be a metric space and let $\emptyset \neq S \subseteq X$, $P \in X$.

Then $d(p,S) = 0 \Leftrightarrow p \in S$ or $p \in \acute{S}$.

Proof:

Let $d(p,S) = 0 \Rightarrow$ Every open set U containing p contains at least one point of $S \Rightarrow p \in S$ or $p \in \acute{S}$

$\Leftarrow$ Let $p \in S$ or $p \in \acute{S}$. T.P $d(p,S) = 0$



$P \in S$ or $P \in \acute{S}$

If $p \in S \Rightarrow d(p,S) = 0$ (see Remark 4.6(2)).

If $p \in \acute{S}$. T.P $d(p,S) = 0$

If not , i.e., let $d(p,S) = \in > 0$ $\Rightarrow B_{\frac{\in}{2}}(p) \cap S = \emptyset$

$$\Rightarrow p \notin \acute{S}(C!)$$

$\therefore d(p,S) = 0$

Corollary 4.1:

If $\emptyset \neq S \subseteq X$ (metric space), then

$\bar{S} = \{x \in X | d(x, s) = 0\}$, where $\bar{S} = S \cup \acute{S}$ (check)

Convergence in Metric Space التقارب في الفضاءات المترية

Definition 4.11:

Let (X, d) be a metric space and let $\langle x_n \rangle$be a sequence in X. We say that $\langle x_n \rangle$converges to a point $x_\circ \in X$, if

$$\forall \epsilon > 0 \ \ \exists k = k(\epsilon) N \text{ s.t } d(x_n, x_\circ) < \epsilon \ \ \forall n > k.$$

The point $x_\circ$ is called the convergence point of $< x_n >$ and we write $\underset{n \to \infty}{}$ $x_\circ$ or limit $x_n = x_\circ$.

- In other word من وجهة نظر اخرى

$x_n \to x_\circ \Leftrightarrow$ Every ball whose center is $x_\circ$ (or Every open set contains $x_\circ$)

contains all but a finite number of the terms of the sequence.

ملاحظة: ان مفهوم التقارب اعلاه لايختلف من حيث الجوهر عن مفهوم التقارب في الاعداد الحقيقية.

Remark 4.7:

Let (X, d) be a metric space and let $x \neq y$ in X, then there exist two open sets V_x and V_y in X s.t

$x \in V_x$, $y \in V_y$ and $V_x \cap V_y = \emptyset$ (check)

Theorem 4.7:

If the sequence $\langle x_n \rangle$is convergent in X, then the convergence point is unique.

Proof:

Let $x_n \to x_\circ$ and $x_n \to y_\circ$ s.t $x_\circ \neq y_\circ$

$\therefore$ by Remark 4.7, $\exists$ two open sets

$V_{x_\circ}$and $V_{y_\circ}$ s.t $x_\circ \in V_{x_\circ}$& $y_\circ \in V_{y_\circ}$ and $V_{x_\circ} \cap V_{y_\circ} = \emptyset$.

Also, $V_{x_\circ}$ and $V_{y_\circ}$ are contains all but a finite number of the term of $< x_n >$ (because $x_n \to x_\circ$, $x_n \to y_\circ$)

$\Rightarrow V_{x_\circ} \cap V_{y_\circ} \neq \emptyset$ $C!$ $\Rightarrow x_\circ = y_\circ$

العلاقة بين مفهومي نقطة التقارب ونقطة التجمع لاي مجموعة موضحة في القضية التالية

Proposition 4.4:

Let X be a metric space, let $\emptyset \neq S \subseteq X$, and let $\langle x_n \rangle$ be a sequence in S.

If $x_n \to x_\circ \in X$, then $x_\circ \in S$ or $x_\circ \in \acute{S}$. Conversely (وبالعكس)

If $x_\circ \in S$ or $x_\circ \in \acute{S}$, then $\exists \langle x_n \rangle$ in S s.t $x_n \to x_\circ$.

Proof:

Suppose that $x_n \to x_\circ$ and $x_\circ \notin S$. T.P $x_\circ \in \acute{S}$

$\because x_n \to x_\circ \Rightarrow \forall r > 0 \ \exists k \in N$ s.t $x_n \in B_r(x_\circ)$ $\forall n > k$.

$\Rightarrow (B_r(x_\circ) \setminus \{x_\circ\}) \cap S \neq \emptyset \;\; \Rightarrow x_\circ \in \acute{S}$. Conversely.

If $x_\circ \in S \Rightarrow$ The Proposition is trivial because we can take $\langle x_n \rangle = \langle x_\circ \rangle$

(Constant seq.) & $\langle x_\circ \rangle \to x_\circ$.

Now, if $x_\circ \notin S$ and $x_\circ \in \acute{S}$

$\forall n \in N$, let $V_n = B_{\frac{1}{n}}(x_\circ)$ in X.

$\because x_\circ \in \acute{S} \Rightarrow S_n = (B_{\frac{1}{n}}(x_\circ) \setminus \{x_\circ\}) \cap S \neq \emptyset$ (if $x \in S_n \Rightarrow$ $d(x, x_\circ) < \frac{1}{n}$)

Let $x_n \in S \;\; \forall n \in N \;\; \Rightarrow \langle x_n \rangle$ is a sequence in S, and

$x_n \to x_\circ$ (by Archimedes property) (check)

Fundamental Sequences (Cauchy Sequences)

Complete Metric Spaces

Definition 4.12:

Let (X, d) be a metric space. A sequence $\langle x_n \rangle$in X is called Cauchy sequence

$\Leftrightarrow \forall \epsilon > 0 \; \exists \; k = k(\epsilon) \in N$ s.t $d(x_n, x_m) < \epsilon \quad \forall n, m > k$.

Proposition 4.5:

Every convergent sequence in a metric space (X, d) is Cauchy sequence.

Proof. (check)

Remark 4.8:

The converse of Prop.4.5 is not true. See the following examples.

Examples 4.12:

1- Let $X = R \setminus \{0\}$ and $d: X \times X \to R$ s.t $d(x, y) = |x - y| \ \forall\, x, y \in X$.

$\therefore$ (X, d) is a metric space & $\langle \frac{1}{n} \rangle$ is Cauchy sequence.

$\Rightarrow$ because $\forall\, \epsilon > 0 \ \exists\, k \in N$ s.t $\frac{1}{k} < \epsilon$

$\therefore \forall\, n > m > k \Rightarrow \frac{1}{n} < \frac{1}{m} < \frac{1}{k} < \epsilon$

$\Rightarrow \left| \frac{1}{n} - \frac{1}{m} \right| < \epsilon \quad \forall n, m > k.$

But $\langle \frac{1}{n} \rangle$ is not convergent in $R \setminus \{0\}$, since $\frac{1}{n} \to 0 \notin R \setminus \{0\}$.

2- Let (Q, d) be a metric space. Then $\exists$ Cauchy sequence

$\langle r_n \rangle$in Q s.t $r_n \to \sqrt{2} \notin Q$

$\therefore$ $\langle r_n \rangle$ not convergent in Q.

Definition 4.13:

A metric space (X, d) is called complete if every Cauchy sequence in X is convergent to a point in X.

Example 4.13:

1- $(R \setminus \{0\}, d)$ and (Q, d) are not complete metric space (see, Ex.4.1.)

2- (R, d) is complete metric space (see, Th.2.7).

Theorem 4.8:

Euclidean metric space R^k is complete. $\forall k \in N$.

Proof:

If $k = 1 \Rightarrow$ The Theorem is true (by Th.2.7)

If $k = 2$

Let $\langle z_m \rangle$ be a Cauchy sequence in R^2 where

$z_m = (x_m, y_m), x_m, y_m \in R$

and let $\epsilon > 0$

$\Rightarrow d(z_m, z_n) < \epsilon, \forall m, n > k.$

$\Rightarrow \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} < \epsilon, \quad \forall m, n > k$

$\Rightarrow (x_m - x_n)^2 + (y_m - y_n)^2 < \epsilon^2, \quad \forall m, n > k$

$\Rightarrow |x_m - x_n| < \epsilon$ and $|y_m - y_n| < \epsilon \quad \forall m, n > k.$

$\Rightarrow \langle x_m \rangle$ and $\langle y_m \rangle$ are Cauchy seq. in R.

$\because R$ complete

$\Rightarrow x_m \to x_\circ \in R$ i.e., $\exists\, k \in N$ s.t $|x_m - x_\circ| < \frac{\epsilon}{2} \quad \forall m > k$

and $y_m \to y_\circ \in R$ i.e., $\exists\, k \in N$ s.t $|y_m - y_\circ| < \frac{\epsilon}{2} \quad \forall m > k$

Now, let $z_\circ = (x_\circ, y_\circ)$, then

$$[d(z_m, z_\circ)]^2 = (x_m - x_\circ)^2 + (y_m - y_\circ)^2 < \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2}.$$

$\Rightarrow d(z_m, z_\circ) < \frac{\epsilon}{\sqrt{2}} < \epsilon \quad \forall m > k \Rightarrow z_m \longrightarrow z_\circ.$

T.P (R^n, d) is complete $\forall\, n \in N$. (if $n = 1$, is true by Th.2.7).

Let $\langle z_m \rangle$ be a Cauchy sequence in R^n, where

$z_m = (z_{1m}, z_{2m}, \cdots, z_{nm}) \in R^n$

$\Rightarrow \forall \epsilon > 0\ \exists k \in N$ s.t $d(z_p, z_q) < \epsilon \quad \forall p, q > k.$

i.e., $\sum_{i=1}^{n}\left(z_{ip}-z_{iq}\right)^2 <\in^2 \quad \forall p,q>k.$

$\Rightarrow (z_{ip}-z_{iq})^{\mathbf{2}} < \epsilon^2 \quad \forall p,q>k$ and $\forall i,\ \ 1\le i\le n.$

$\Rightarrow \left|z_{ip}-z_{iq}\right| < \epsilon \quad \forall p,q>k$ and $\forall i,\ \ 1\le i\le n.$

$\therefore \langle z_{im}\rangle$ is Cauchy in $R \quad \forall i,\ \ 1\le i\le n.$

$\because R$ is complete

$\Rightarrow z_{im}\to \bar{z}_i$

Define , $\bar{z}=(\bar{z}_1,\bar{z}_2,\cdots,\bar{z}_m)$

T.P $z_m\to\bar{z}$

$\{\text{i. e., to prove } \forall\epsilon>0\ \ \exists k\in N\ \text{ s. t }\ d(z_p,\bar{z})<\epsilon \quad \forall p>k\}$

$[d(z_p,\bar{z})]^{\mathbf{2}}=\sum_{i=1}^{n}\left(z_{ip}-\bar{z}_i\right)^2 < \frac{\epsilon^2}{n},\quad p>\max(k_i).$

$\Rightarrow d(z_p,\bar{z}) < \frac{\epsilon}{\sqrt{n}} < \epsilon \quad \forall p>k$ where $k>\max(k_i).$

$\therefore z_m\to\bar{z}$

ملاحظة : كل فضاء غير كامل يمكن ان يغمر (Embedding) في فضاء متري كامل وذلك باضافة

نقاط التقارب للمتتابعات الكوشية في الفضاء للحصول على فضاء جديد كامل . كما هو الحال

عند اضافة نقاط التقارب للمتتابعات الكوشية في الاعداد النسبية لاجل الحصول على الاعداد

الحقيقية .