Metric Spaces   الفضاءات المترية

مقدمة : لنظام الاعداد الحقيقية نوعان من الخواص، النوع الاول هو خواص جبرية تتعلق بالجمع والطرح والضرب والقسمة واستخراج الجذور. أما النوع الثاني من الخواص فهي التي تتعلق بمفهوم البعد (المسافة) بين عددين ومفهوم التقارب ، ويدعى هذا النوع من الخواص بالخواص التبولوجية أو المترية ، وموضوع التحليل الرياضي يتعلق بدراسة هذا النوع من الخواص.

Definition 4.1:

Let X be a non-empty set. A function $d : X \times X \rightarrow R$ is said to be a metric (or distance function) on X if d satisfies the following conditions :

1. $d(x, y) \geq o\ , \quad \forall\, x, y \in X.$
2. $d(x, y) = o \iff x = y.$
3. $d(x, y) = d(y, x) \quad \forall\, x, y \in X$ (Symmetry)
4. $d(x, y) \leq d(x, z) + d(z, y) \quad \forall\, x, y, z \in X$ (Triangle inequality)

If d is a metric on X, then (X, d) is called a metric space, and $d(x, y)$ is called the distance between x and y.

Example 4.1:

1- Let $X = R$ and $d: R \times R \rightarrow R$ a function defined by

$$d(x, y) = |x - y|, \quad \forall\, x, y \in R.$$

Then d is a metric on R called absolute value metric (or usual metric) on R. And (R, d) is a metric space [see, Def. 1.10].

2- Let $X \neq \phi$ be any set and $d : X \times X \rightarrow R$ defined by

$$d(x,y)=\begin{cases}1 & if\ x\neq y\\ 0 & if\ x=y\end{cases}$$

Then d is a metric on X.

Proof: 1, 2, 3 are clear from definition of d.

1- $d(x,y)=\begin{cases}1 & if\ x\neq y\\ 0 & if\ x=y\end{cases}\geq 0,$

2- $d(x,y)=\begin{cases}1 & if\ x\neq y\\ 0 & if\ x=y\end{cases}=0\ \ if\ x=y$???

3- $d(y,x)=\begin{cases}1 & if\ y\neq x\\ 0 & if\ y=x\end{cases}=d(x,y)$??????

For condition (4)

الحالات $d(x,y)\leq d(x,z)+d(z,y)$ الطرف الايمن ياخذ القيم 0، 1، 2

(أعلى قيمة) 1- $x=y=z$ 0 0 0T

2- $x\neq y; y\neq z$ 1 1 1 T

3- $x\neq y; y=z$ 1 1 0 T

4- $x=y, y\neq z$ 0 1 1 T

Then $\boldsymbol{d}$ is a metric function.

$\therefore$ $(\boldsymbol{X},\boldsymbol{d})$ **is a metric space and called discrete space.**

ملاحظة : لا بد من التاكد بان الفضاء المتري هو ليس المجموعة X لوحدها بل X مع دالة البعد

d ، حيث يمكن أن نجعل من المجموعة X فضاءاً متريا باكثر من طريقة واحدة وذلك باعطاء صيغ مختلفة ل d

3- Let (X,d) be a metric space and let

$d_1(x,y) = K\ d(x,y), \quad K > 0$. Prove that d_1 is a metric on X. (check)

4- Let $R^2 = \{(x,y) |\ x, y\ \in R\}$ and let $d: R^2 \times R^2 \to R$ be a function s.t

$$d(x_1,y_1),(x_2,y_2) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

Then d is a metric on R^2.

Proof:

Let $P_1, P_2, P_3\ \in\ R^2$ s.t $P_i = (x_i, y_i), \ \ i = 1,2,3$.

1- $d\ (P_1,P_2) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\ \geq 0 \quad \forall\ P_1, P_2\ \in R^2$.

2- $d\ (P_1,P_2)\ = 0 \iff \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = 0 \iff$

$(x_2-x_1)^2 = 0\ \forall\ (y_2-y_1)^2 = 0 \iff x_2 = x_1\ \ \forall\ y_2 = y_1$

$\iff\ (x_1,y_1) = (x_2,y_2) \iff P_1 = P_2$.

3- $d\ (P_1,P_2)\ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

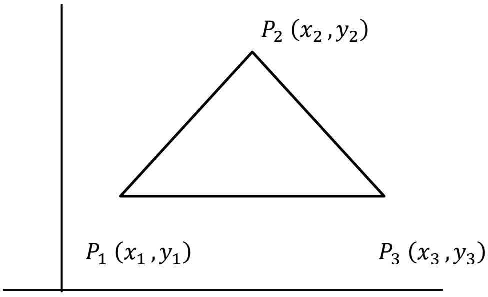
$= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$= d\big((x_2,y_2),(x_1,y_1)\big) = d\ (P_2,P_1)$

$(x_1-x_2)^2 = 1.(x_1-x_2)^2 = (-1)^2(x_1-x_2)^2 = (-1(x_1-x_2))^2 =$

$(x_2-x_1)^2$. م

4- $d\ (P_1\ , P_3) \leq d\ (P_1\ , P_2) + d(P_2\ , P_3)$



مجموع أي ضلعين في مثلث
أكبر من الضلع الثالث

$\Rightarrow d$ is a metric on R^2

$\therefore\ (R^2, d)$ is a metric space

5- Let $(x_1, y_1), (x_2, y_2) \in R^2$. Which of the following define a metric on R^2.

(check)

a. $d\left((x_1, y_1), (x_2, y_2)\right) = |x_1 - x_2| + |y_1 - y_2|$.

b. $d\left((x_1, y_1), (x_2, y_2)\right) = |x_1 - x_1| + |y_2 - y_2|$.

c. $d\left((x_1, y_1), (x_2, y_2)\right) = \max\ \{|x_1 - x_2|\ , |y_1 - y_2|\}$.

d. $d\left((x_1, y_1), (x_2, y_2)\right) = \min\ \{|x_1 - x_2|\ , |y_1 - y_2|\}$.

Remark 4.1:

1- Let (X, d) be a metric space, and $x, y, z \in X$, then

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

Proof:

By triangle inequality we get that

$d(x, z) \leq d(x, y) + d(y, z) \Rightarrow d(x, z) - d(y, z) \leq d(x, y)$
(1)

$d(y, z) \leq d(y, x) + d(x, z) \Rightarrow -d(y, x) \leq d(x, z) - d(y, z)$
(2)

$\because\ d(x,y) = d(y,x)$. Then (1) + (2) $\Rightarrow \lceil d(x,z) - d(y,z) \rceil \leq d(x,y)$.

2- Let (X,d) be a metric space, and $x_1, x_2, \ldots\ x_n \in X$, then

$$d(x_1,x_n) \leq d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{n-1},x_n)$$

تسمى هذه المتراجحة بالمتراجحة المضلعية وتبرهن باستخدام الاستقراء الرياضي على n وباستخدام المتراجحة المثلثية.

Definition 4.2:

Let (X,d) be a metric space and $\phi \neq S \subseteq X$. If

$$d_s : S \times S \to R \text{ s.t } d_s(x,y) = d(x,y)\ \forall (x,y) \in S.$$

then we say that (S,d_s) is a subspace of (X,d).

Proposition 4.1:

$\forall\ \ a,b \in R,\ \ a \geq 0\ \ \forall\ \ b \geq 0$ we get

$$(ab)^{1/2} \leq \frac{a+b}{2}$$

المعدل الحسابي   المعدل

الهندسي

للعددين

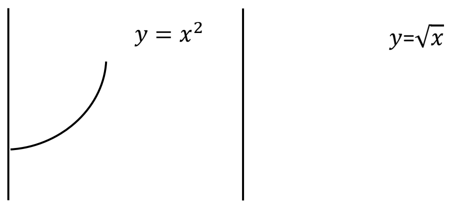
b,a

أي ، المعدل الحسابي أكبر من أو يساوي المعدل الهندسي

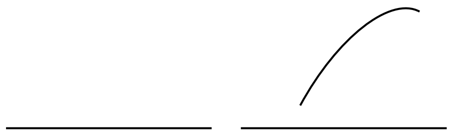
Proof:

دالة متزايدة

$$ab \leq \frac{a^2+2ab+b^2}{4}$$



$\Rightarrow \; 4ab \le a^2 + 2ab + b^2$



$\Rightarrow \quad 0 \le a^2 - 2ab + b^2 = (a-b)^2$

Cauchy-Schwarz inequality **متباينة كوشي- شوارز**

Let $x_1, x_2, \dots x_n$ and $y_1, y_2, \dots y_n$ be real numbers, then

$$|x_1y_1 + x_2y_2 + \dots + x_ny_n| \le \sqrt{x_1^2 + \dots + x_n^2} \,.\, \sqrt{y_1^2 + \dots + y_n^2}$$

i.e.,$\sum_{i=1}^{n} |x_iy_i| \le \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}} \;\; \forall\, x_i \in R, \forall\, y_i \in R \;\; 1 \le i \le n$

where n is a positive integer

Proof :

Let $X = (x_1, x_2, \dots x_n) \ne 0$(vector) $\forall \;\; Y = (y_1, y_2, \dots y_n) \ne 0$

$\Rightarrow x_i \ne 0 \;\; \forall\, y_i \ne 0$ for at least or $1 \le i \le n$.

Assume $a_i = \frac{x_i^2}{\sum_{j=1}^{n} x_j^2} \ge 0$

$$\forall i, \quad 1 \le i \le 2$$

$b_i = \frac{y_i^2}{\sum_{j=1}^{n} y_j^2} \ge 0$

$\Rightarrow \sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} b_i = 1$.

By Prop.4.1, we get that

$$(a_i b_i)^{\frac{1}{2}} \leq \frac{a_i+b_i}{2} \quad \forall\, i$$

$$\Rightarrow \frac{|x_i|}{\left(\sum x_j^2\right)^{1/2}} \cdot \frac{|y_i|}{\left(\sum y_j^2\right)^{1/2}} \leq \frac{x_i^2|\sum x_j^2+y_i^2|\sum y_j^2}{2} \qquad (\sqrt{x_i^2}\ |x_i|\ )$$

$$\because\ |ab|^{\frac{1}{2}} = a^{\frac{1}{2}} . b^{\frac{1}{2}}. \text{ Then}$$

$$\sum_{i=1}^{n} \frac{|x_i|}{\left(\sum x_j^2\right)^{1/2}} \cdot \frac{|y_i|}{\left(\sum y_j^2\right)^{1/2}} \leq \sum_{i=1}^{n} \frac{x_i^2}{\sum x_j^2} + \frac{y_i^2}{\sum y_j^2} / 2$$

$$\Rightarrow \frac{\sum_{i=1}^{n} |x_i y_i|}{\left(\sum x_j^2\right)^{1/2} \left(\sum y_j^2\right)^{1/2}} \leq \frac{1}{2} + \frac{1}{2} = 1$$

$$\Rightarrow \sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}} \quad \left(\sum_{j=1}^{n} y_j^2\right)^{\frac{1}{2}}.$$

Now, if $X = 0$ or $Y = 0$ $\quad \Rightarrow x_1 = x_2 = \ldots = x_n = 0$ or

$y_1 = y_2 = \ldots = y_n = 0$

and the result it is clear.

Minkowski Inequality متراجحة منكوفسكي

Let $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ be real numbers, then

$$\left(\sum_{i=1}^{n} (a_i + b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} {a_i}^2\right)^{1/2} + \left(\sum_{i=1}^{n} {b_i}^2\right)^{1/2}$$

Proof:

$$\sum_{i=1}^{n}(a_i + b_i)^2 = \sum_{i=1}^{n} {a_i}^2 + 2\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2$$

$$\leq \sum_{i=1}^{n} {a_i}^2 + 2\sum_{i=1}^{n}|a_i b_i| + \sum_{i=1}^{n} b_i^2$$

$$\leq \sum_{i=1}^{n} {a_i}^2 + 2\left[\left(\sum_{i=1}^{n} {a_i}^2\right)^{1/2} \left(\sum_{i=1}^{n} {b_i}^2\right)^{1/2}\right] + \sum_{i=1}^{n} b_i^2$$

$$= \left[\left(\sum_{i=1}^{n} {a_i}^2\right)^{1/2} + \left(\sum_{i=1}^{n} {b_i}^2\right)^{1/2}\right]^2$$

وباخذ الجذر للطرفين نحصل على متراجحة منكوفسكي

Example 4.2:

Let $R^2 = \{X = (x_1, x_2, \dots x_n) |\, x_i \in R, 1 \leq i \leq n\}$

If $X = (x_1, x_2, \dots x_n) \in R^n$ and $Y = (y_1, y_2, \dots y_n) \in R^n$

Define $d: R^n \times R^n \to R$ by follows

$d(X,Y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}$.
$\forall\, X, Y \in R^2$

Prove that (R^2, d) is a metric space.

Proof:

(1), (2) ∀ (3) are hold by definition of d (check)

Now, let $X = (x_1, x_2, \dots x_n) \in R^n$

$Y = (y_1, y_2, \dots y_n) \in R^n$

$Z = (z_1, z_2, \dots z_n) \in R^n$

T.P $d(X,Y) \leq d(X,Z) + d(Z,Y)$

$\because x_i - y_i = x_i - z_i + z_i - y_i$

$\therefore d(X,Y) = \left(\sum_{i=1}^{n} (x_i - z_i + z_i - y_i)^2\right)^{1/2}$

$\leq \left(\sum_{i=1}^{n} (x_i - z_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n} (z_i - y_i)^2\right)^{1/2}$ (by Minkowski inquality)

$= d(X,Z) + d(Z,Y)$