Density of Irrational Numbers

Proposition 1.3: Let $r \in Q$ and $s \in Q^C$, then

1. $r + s \in Q^C$
2. If $r \neq 0$, then $rs \in Q^C$.

Proof: (1) Let $r + s \notin Q^C \Rightarrow r + s \in Q$, since Q is field, $r \in Q \Rightarrow -r \in Q$, then $(r + s) - r = s \in Q \Rightarrow C!$ (because s $\in Q^C$).

(2) Let $rs \notin Q^C \Rightarrow rs \in Q$, since Q is field, and $r \neq 0 \Rightarrow (rs).\frac{1}{r} = s \in Q \Rightarrow C!$.

Theorem 1.5: if $a, b \in R$ such that $a < b$, then there exists $s \in Q^C$ such that $a < s < b$.

Or ((Between any two real numbers there is at least one irrational number)).

Proof: suppose that this theorem is not true.

$\therefore\ \forall s \in R$ s.t $a < s < b \Rightarrow s \in Q \Rightarrow a + \sqrt{2} < s + \sqrt{2} < b + \sqrt{2}$,

this is contradiction with density of rational numbers.

$\therefore\ a < s < b, s \in Q^C$. ■

Corollary 1.3: Let $a < b$, prove that there exists infinitely many of irrational numbers between a, b.

Proof: (check).

The Absolute Value Function

Definition 1.9: The absolute value is a function $|.|: R \to R^+ \cup \{0\}$ defined by

$$|x| = \sqrt{x^2} = \begin{cases} x, \text{if } x \geq 0 \\ -x, \text{if } x < 0 \end{cases}$$

$|x|$ is called the absolute value of x and $|.|$ is called the absolute value function.

❖ Properties of the absolute value function

Theorem 1.6: The function $|.|$ on R satisfying the following properties:

1. $|x| \geq 0, \forall x \in R,$
2. $|x| \geq 0 \iff x = 0,$
3. $|x.y| = |x|.|y|, \forall x \in R,$
4. $|x + y| \leq |x| + |y|, \forall x, y \in R,$
5. $|x - y| \geq \big||x| - |y|\big|, \forall x, y \in R.$

Proof: (check).

We shall use the absolute value function to define the distance between two real numbers as follows:

Definition 1.10: Let $d: R \times R \to R$ be a function define by

$d(x, y) = |x - y|, \forall\, x, y \in R.$ $d(x, y)$ is called the distance between x and y.

The function d is satisfies the following conditions:

1. $d(x, y) \geq 0, \forall x, y \in R.$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x), \forall x, y \in R$
4. $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in R$

Proof: $\forall x, y, z \in R$, we have

1. $\because\ d(x, y) = |x - y| \geq 0.$
2. $d(x, y) = 0 \iff |x - y| = 0 \iff x - y = 0 \iff x = y$

$$\Rightarrow d(x, y) = 0 \iff x = y$$

3. $d(x, y) = |x - y| = |-(y - x)| = |-1||y - x|$

$$= |y - x| = d(y, x)$$

4. $d(x,y) = |x-y| = |x-z+z-y| \leq |x-z| + |z-y|$

$$= d(x,z) + d(z,y). \quad \blacksquare$$

General Information's

Theorem 1.7: Let $\varphi \neq A \subset R$ and $\sup(A)$ exist, prove that

$\exists\, x \in A \text{ s.t } \sup(\text{A}) - \epsilon < x \leq \sup(A)$.

Proof: (check).

Corollary 1.4: prove that the natural numbers set N is unbounded set.

Corollary 1.5: Let A and B be a non-empty subset of R and let $a < b, \forall a \in A, \forall b \in B$. Prove that $\sup(A) \leq \inf(\text{B})$.

The Extended Real Numbers System

Definition 1.11: The Extended Real Numbers System consists of real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in R, and define $+\infty < x < -\infty, \forall x \in R$.

Let R^* denoted the extended real numbers systems, then $+\infty$ is an upper bounded of every subset of R^*, and every non-empty subset has a least upper bounded.

Example 1.8: Let $\varphi \neq E \subset R$ which is not bounded above in R, then $\inf(E) = +\infty$ in R^*. And, if $\varphi \neq E \subset R$ which is not bounded below in R, then $\inf(E) = -\infty$ in R^*.

Remark: R^* does not form a field.

Some properties:

a) If $x \in R$, then $x + \infty = +\infty, x - \infty = -\infty, \frac{x}{+\infty} = \frac{x}{-\infty} = 0$.

b) If $x > 0$, then $x.(+\infty) = +\infty, x.(-\infty) = -\infty$.

c) If $x < 0$, then $x.(+\infty) = -\infty$, $x.(-\infty) = -\infty$.