The relation between the field of real numbers and

The field of rational numbers

Proposition 1.1: Every ordered field contains a subfield like the field of rational numbers.

Proof: Let $(F, +, .)$ be any ordered field.

$\therefore\ 0, 1 \in F$ (0 is additive identity and 1 is multiplicative identity), then we see that

$$\underbrace{1 + 1 + 1 + \cdots + 1}_{(n-\text{times})} = n.1\ (n \in Z^+) \Rightarrow n.1 \neq 0 \text{ (why?)}$$

Thus, if $n.1 = 0, \forall\, n \in Z^+$. So, If it is not, *i.e.*, let $n.1 = 0$ for some $n \in Z^+$, and let K be the least positive integer such that

$$\underbrace{1 + 1 + 1 + \cdots + 1}_{(K-\text{times})} = k.1 = 0$$

Clearly that $k > 1$ (since $1 > 0$) and $(k - 1).1 > 0$ (why?)

$\Rightarrow 0 < (k - 1).1 < k.1 = 0 \Rightarrow 0 < 0 \ \Rightarrow C!$,

thus $n.1 = 0 \Leftrightarrow n = 0\ (n \in Z^+)$.

From the above remarks we see that F contains elements of kind

$$n.1 = n, \ (n \in Z^+),$$

and

$$n.1 = 0 \Leftrightarrow n = 0\,,$$

also $n.1 = m.1 \Leftrightarrow n = m$.

$\because\ F$ is a field $\Rightarrow$ F contains $-n.1$ where

$$\underbrace{(-1) + (-1) + (-1) + \cdots + (-1)}_{(-1)(n-\text{times})} = -n.1$$

Thus, F contains a copy of Z.

$\because$ F is a field $\Rightarrow \forall 0 \neq n \in Z \Rightarrow \frac{1}{n} \in F$,

therefore F contains a copy of Q. ■

Proposition 1.2: The equation $x^2 = 2$ has no solution in Q.

Proof: Assume that $x^2 = 2$ has a solution in Q, say y

$\therefore y \in Q \Rightarrow y = \frac{a}{b}, b \neq 0 \ (a, b \in Z)$ and $y^2 = 2 \Rightarrow a^2 = 2\,b^2,\ a^2 > 0$

Suppose that a, b is positive number and the greatest common factor between them is 1. Then we have the following cases:

1. a and b are odd $\Rightarrow a^2 = 2\,b^2 \Rightarrow C!$
2. a is odd, b is even. When b is even then $b = 2c \ (c \in N)$

$\because a^2 = 2\,b^2 \Rightarrow a^2 = 8\,c^2 \Rightarrow C!$

3. a is even, b is odd, then $a = 2d \ (d \in N)$

$\because a^2 = 2\,b^2 \Rightarrow 4d^2 = 2\,b^2 \Rightarrow 2d^2 = b^2 \Rightarrow C!$

From all above we get that $x^2 = 2$ has no solution in the rational number Q ■

Theorem 1.1: The equation $x^2 = 2$ has a unique positive real root.

Proof: Let $S = \{x \in Q | x > 0 \text{ and } x^2 < 2\}$

$\because 1 \in S \Rightarrow S \neq \varphi$, and S is bounded above (since 2,3, ..., is an upper bound of S), then by completeness property of R, S has least upper bound in R.

Let $\sup(\text{S}) = y_0,\ y_0 \in R$. Clearly that $y_0 > 0$ (by Def. of S).

Now, claim that $y_0^2 = 2$, if not (i.e.$y_0^2 \neq 2$), then $y_0^2 < 2$ or $y_0^2 > 2$.

1. If $y_0^2 < 2$, choose h s.t $0 < h < 1$

$\therefore (y_0 + h)^2 = y_0^2 + 2y_0h + h^2 = y_0^2 + h(2y_0 + h) < y_0^2 + h(2y_0 + 1)$

Also, let h satisfy the following condition

$$h < \frac{2 - y_0^2}{2y_0 + 1} \Rightarrow y_0^2 + h(2y_0 + 1) < 2$$

$$\Rightarrow (y_0 + h)^2 < y_0^2 + h(2y_0 + 1) < 2$$

$\because$ $(y_0 + h)^2 < 2$, then $(y_0 + h) \in S$, when $y_0 < y_0 + h \Rightarrow y_0$ is not upper bound of $S \Rightarrow C!\,(y_0 = \sup(S))$.

2. If $y_0^2 > 2$, choose k s.t $0 < k < 1$

$\therefore (y_0 - k)^2 = y_0^2 - 2y_0k + k^2 = y_0^2 - k(2y_0 - k) > y_0^2 - k(2y_0 + 1)$

Also, let k satisfy the following condition

$$k > \frac{y_0^2 - 2}{2y_0 + 1} \Rightarrow k(2y_0 + 1) < y_0^2 - 2 \Rightarrow -k(2y_0 + 1) > 2 - y_0^2$$

$$\Rightarrow y_0^2 - k(2y_0 + 1) > 2 \Rightarrow (y_0 + k)^2 > y_0^2 - k(2y_0 + 1) > 2$$

$\Rightarrow (y_0 - k)^2 > 2$, then $(y_0 - k)$is an upper bound of S, thus $y_0 > y_0 - k$ (because y_0 least upper bound of S) $\Rightarrow C!\,(y_0 - k < y_0)$.

From (1) and (2) $\Rightarrow y_0^2 = 2$.

Here to prove that y_0 is unique. Let $\exists\, z \in R$ s.t $z \neq y_0$ and $z^2 = 2$

$\because z \neq y_0$, then either $z > y_0 \Rightarrow 2 = z^2 > y_0^2 = 2 \Rightarrow 2 > 2 \Rightarrow C!$

or $z < y_0 \Rightarrow 2 = z^2 < y_0^2 = 2 \Rightarrow 2 < 2 \Rightarrow C!$

Thus $z = y_0$ and y_0 is the only one positive real root to $x^2 = 2$ ■

Remark:

1. From above Q is a proper subfield of R (i.e.$Q \subsetneq R$) because$\sqrt{2} \in R$, but $\sqrt{2} \notin Q$.
2. Q is not complete, because $S = \{x \in Q | x > 0 \text{ and } x^2 < 2\} \subset Q, S \neq \varphi$ $(1 \in S)$

S is bounded above and $\sup(S) = \sqrt{2} \notin Q$, then S has no least upper bound in Q, thus Q is not complete field.

$\mathbb{Q}_2$: prove that $x^2 = 3$ has no solution in Q. (Check)
$\mathbb{Q}_3$: prove that $x^2 = 3$ has a unique positive real root. (Check)

Theorem 1.2: For any positive real number a and for any $n \in Z^+, \exists!$ positive real number satisfies the following $x^n = a$ and denoted this unique number by $\sqrt[n]{a}$ (or $a^{1/n}$).

Proof: The proof is similar to proof of (Theorem 1.1).

$\mathbb{Q}_4$: Let a, b be tow positive real numbers and $n \in Z^+$ prove that $(a, b)^{\frac{1}{2}} = a^{\frac{1}{2}}.b^{\frac{1}{2}}$ (Check)

Theorem 1.3: ***(Archimedes property)***

For any real numbers a, b and $a > 0$, there is a positive integer number n s.t $na > b$.

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Let any real numbers a, b ann if there is a positive integer number n such that $na \leq b$, then, $a < 0$

and $a > 0$, there is a positive integer number n s.t $na > b$.

Proof: Let $S = \{ak | k \in N\} \subset R, S \neq \varphi$ (becauce $1.a = a \in S$

$\therefore$ by completeness of R, S has least upper bound. Let $y = \sup(S), (y \in R)$

$\therefore a > 0 \Rightarrow y - a < y \Rightarrow y - a$ is not upper bound of S. Hence, $\exists$ element $ma \in S$ (for some $m > 0, m \in Z^+$) s.t $y - a < ma \Rightarrow ma + a > y \Rightarrow (m+1)a > y$

But, $(m+1)a \in S, (m+1 \in N) \Rightarrow C!$, because y is an upper bound of S. Then $na > b$ ■

Corollary 1.1: For any positive real number ϵ, there is a positive integer n such that $\frac{1}{n} < \epsilon$. (i. e.) $\forall \epsilon > 0, \exists\, n \in N$ s. t $\frac{1}{n} < \epsilon$.

Proof: Take $b = 1$ and $a = \epsilon$, then by (Theorem 1.3), we get that $\frac{1}{n} < \epsilon$.

Density of Rational Numbers

Theorem 1.4: If $a, b \in R$ such that $a < b$, then there exists $r \in Q$ such that $a < r < b$.

Or ((Between any two real numbers there is at least one rational number)).

Proof:

Case 1: Let $0 < a < b$, and $b - a > 1$.

and

Let $S = \{n \in N | \, n = n.1 > a\}$

$\therefore S \neq \varphi$ (by Archimedes property) (How?)

Let k be the smallest positive integer in S [$\because \varphi \neq S \subset N$ and N is well-ordered set, then by (Def. 1.6) S has a smallest element]

$$\Rightarrow \left.\begin{matrix} k-1 \leq a < k \\ b-a > 1 \end{matrix}\right\} \Rightarrow a < k < b \text{ (How?)}$$

In this case k is the rational number between a and b (k is an integer number).

Now, if

$$0 < a < b, \text{and } 0 < b-a < 1.$$

when $(b-a) > 0$, then by (Archimedes property), $\exists\, n \in N$ s.t $n(b-a) = nb - na > 1$

$\therefore$ From case (1), $\exists\, k \in N$ s.t $na < k < nb \Rightarrow a < \frac{k}{n} < b$ and hence $\frac{k}{n}$ is rational number.

Case 2: $a < 0 < b$, in this case 0 is the rational number between a, b.

Case 3: $a < b < 0 \Rightarrow\ 0 < -b < -a$

By case (1), $\exists\, k \in Q$ s.t $-b < r < -a \Rightarrow a < -r < b$. ■

Corollary 1.2: Let $a < b$, prove that there exists infinitely many of rational numbers between a, b.

Proof: (check).