6.3 Linearity of Riemann Integral (خطية تكامل ريمان)

Theorem 6.13: If f is R-integrable on $[a,b]$ and k is any constant, then kf is R-integrable on $[a,b]$ and $k\int_a^b f = \int_a^b kf$.

Proof: Let $\epsilon > 0$, since f is R-integrable on $[a,b]$.

$\Rightarrow \exists$ a partition on $[a,b]$

$\Rightarrow \underline{\int_a^b} f - \epsilon < \underline{R}(f,\rho)$ and $\overline{\int_a^b} f + \epsilon > \overline{R}(f,\rho)$.

$\because \underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f \Rightarrow \int_a^b f - \epsilon < \underline{R}(f,\rho)$,

and

$\int_a^b f + \epsilon > \overline{R}(f,\rho)$.

Case (1): If $k > 0$

$\Rightarrow k\underline{R}(f,\rho) = \underline{R}(kf,\rho)$.

and

$k\overline{R}(f,\rho) = \overline{R}(kf,\rho)$.

$\therefore k\int_a^b f - k\epsilon < \underline{R}(kf,\rho)$ and $k\int_a^b f + k\epsilon < \overline{R}(kf,\rho)$.

But $k\int_a^b f - k\epsilon < \underline{R}(kf,\rho) \leq \underline{\int_a^b} kf$

and

$\overline{\int_a^b} kf \leq \overline{R}(kf,\rho)$

$\therefore k\int_a^b f - k\epsilon < \underline{\int_a^b} kf \leq \overline{\int_a^b} kf < k\int_a^b f + k\epsilon$.

$\because \epsilon$ is an arbitrary, then,

$k\int_a^b f \leq \underline{\int_a^b} kf \leq \overline{\int_a^b} kf \leq k\int_a^b f$

$\Rightarrow \underline{\int_a^b} kf = \overline{\int_a^b} kf = k\int_a^b f$

$\Longrightarrow kf$ is R-integrable on $[a,b]$ and $k\int_a^b f = \int_a^b kf$.

Case (2): If $k < 0$.

$\Longrightarrow k\underline{R}(f,\rho) = \overline{R}(kf,\rho)$ and $k\overline{R}(f,\rho) = \underline{R}(kf,\rho)$.

$\therefore k\int_a^b f + k\epsilon < \underline{R}(kf,\rho) \leq \int_{\underline{a}}^b kf$.

and

$$\int_a^{\overline{b}} kf \leq \overline{R}(kf,\rho) < k\int_a^b f - k\epsilon$$

$\therefore k\int_a^b f + k\epsilon < \int_{\underline{a}}^b kf \leq \int_a^{\overline{b}} kf < k\int_a^b f - k\epsilon.$

$$k\int_a^b f \leq \int_{\underline{a}}^b kf \leq \int_a^{\overline{b}} kf \leq k\int_a^b f$$

$$\Longrightarrow \int_{\underline{a}}^b kf = \int_a^{\overline{b}} kf = k\int_a^b f.$$

$\Longrightarrow kf$ is R-integrable on $[a,b]$ and $k\int_a^b f = \int_a^b kf$.

Case (3): If $k = 0$.

$\Longrightarrow$ each side is zero.

Theorem 6.14: If f_1 and f_2 are R-integrable on $[a,b]$, then $f_1 + f_2$ is R-integrable on $[a,b]$ and $\int_a^b (f_1 + f_2) = \int_a^b f_1 + \int_a^b f_2$.

Proof: let $\epsilon > 0$, since f_1 and f_2 are R-integrable on $[a,b]$.

$\Longrightarrow \exists$ partitions ρ_1 and ρ_2 on $[a,b]$ such that

$$\int_{\underline{a}}^b f_1 - \epsilon < \underline{R}(f_1,\rho_1) \text{ and } \int_a^{\overline{b}} f_1 + \epsilon > \overline{R}(f_1,\rho_1).$$

$$\int_{\underline{a}}^b f_2 - \epsilon < \underline{R}(f_2,\rho_2) \text{ and } \int_a^{\overline{b}} f_2 + \epsilon > \overline{R}(f_2,\rho_2)$$

Let $\rho = \rho_1 \cup \rho_2$. Let m_i and $m"_i$ be infimum of f_1 and f_2 respectively on the segment $[x_{i-1}, x_i]$ of ρ.

$\therefore m'_i + m"_i \leq f_1(x) + f_2(x), \quad \forall x \in [x_{i-1}, x_i].$

$\therefore \underline{R}(f_1,\rho)+\underline{R}(f_2,\rho)\le \underline{R}(f_1+f_2,\rho)$

Also, $\overline{R}(f_1,\rho)+\overline{R}(f_2,\rho)\ge \overline{R}(f_1+f_2,\rho)$.

$$\underline{\int_a^b} f_1+\underline{\int_a^b} f_2-2\,\epsilon<\underline{R}(f_1+f_2,\rho)$$

And

$$\overline{\int_a^b} f_1+\overline{\int_a^b} f_2+2\,\epsilon>\overline{R}(f_1+f_2,\rho)$$

But $\underline{R}(f_1+f_2,\rho)\le \underline{\int_a^b}(f_1+f_2)$

and

$$\overline{\int_a^b}(f_1+f_2)\le \overline{R}(f_1+f_2,\rho).$$

$$\therefore \underline{\int_a^b} f_1+\underline{\int_a^b} f_2-2\,\epsilon<\underline{\int_a^b}(f_1+f_2)$$

$$\le \overline{\int_a^b}(f_1+f_2)<\overline{\int_a^b} f_1+\overline{\int_a^b} f_2+2\,\epsilon.$$

Since ϵ is an arbitrary and f_1 and f_2 are R-integrable on $[a,b]$, then f_1+f_2 is R-integrable on $[a,b]$ and $\int_a^b(f_1+f_2)=\int_a^b f_1+\int_a^b f_2$.

Theorem 6.15: If f is R-integrable on $[a,b]$ and $a<c<b$, then f is R-integrable on $[a,c]$, $[c,b]$ and $\int_a^c f+\int_c^b f=\int_a^b f$.

Proof: Let $\epsilon>0$, since f is R-integrable on $[a,b]$.

$\Longrightarrow \exists$ a partition ρ on $[a,b]$ such that

$$\int_a^b f-\epsilon<\underline{R}(f,\rho) \text{ and } \overline{R}(f,\rho)<\int_a^b f+\epsilon.$$

Let $\rho_1=\rho\cap[a,c]$ be a partition on $[a,c]$.

$\rho_2=\rho\cap[c,b]$ be a partition on $[c,b]$.

$$\underline{R}(f,\rho)=\underline{R}(f,\rho_1)+\underline{R}(f,\rho_2)$$

and

$$\overline{R}(f,\rho)=\overline{R}(f,\rho_1)+\overline{R}(f,\rho_2)$$. Hence

$$\int_a^b f - \epsilon < \underline{R}(f,\rho)e$$

$$\Rightarrow \int_a^b f - \epsilon < \underline{R}(f,\rho_1) + \underline{R}(f,\rho_2) \leq \int_{\underline{a}}^{c} f + \int_{\underline{c}}^{b} f \qquad (1)$$

$$\overline{R}(f,\rho) < \int_a^b f + \epsilon.$$

$$\Rightarrow \int_a^{\overline{c}} f + \int_c^{\overline{b}} f \leq \overline{R}(f,\rho_1) + \overline{R}(f,\rho_2) < \int_a^b f + \epsilon \qquad (2)$$

From (1)and (2) we get

$$\int_a^b f - \epsilon < \int_{\underline{a}}^{c} f + \int_{\underline{c}}^{b} f \leq \int_a^{\overline{c}} f + \int_c^{\overline{b}} f < \int_a^b f + \epsilon$$

Since ϵ is an arbitrary

$$\Rightarrow \int_a^b f < \int_{\underline{a}}^{c} f + \int_{\underline{c}}^{b} f \leq \int_a^{\overline{c}} f + \int_c^{\overline{b}} f < \int_a^b f.$$

$$\Rightarrow \int_{\underline{a}}^{c} f + \int_{\underline{c}}^{b} f = \int_a^{\overline{c}} f + \int_c^{\overline{b}} f.$$

$$\Rightarrow \int_{\underline{a}}^{c} f = \int_a^{\overline{c}} f \text{ and } \int_{\underline{c}}^{b} f = \int_c^{\overline{b}} f.$$

then f is R-integrable on $[a,c]$.

And on $[c,b]$ *and* $\int_a^c f + \int_c^b f = \int_a^b f$.

Corollary 6.16: If f is R-integrable on $[a,b]$ and $[c,d] \subset [a,b]$, then f is R-integrable on $[c,d]$.

Proof: Since $a < c < b$ and f is R-integrable on $[a,b]$.

$\Rightarrow f$ is R-integrable on $[a,c]$and $[c,b]$.

But $c < d < b$, then f is R-integrable on $[c,d]$and $[d,b]$

$\Rightarrow f$ is R-integrable on $[c,d]$.

Theorem 6.17: If f is R-integrable on $[a,c]$ and $[c,b]$, then f is R-integrable on $[a,b]$.

Proof: Let $\epsilon > 0$, since f is R-integrable on $[a,c]$ and $[c,b]$, then $\exists$ partitions ρ_1 and ρ_2 on $[a,c]$ and $[c,b]$ respectively such that

$\int_a^c f - \epsilon < \underline{R}(f, \rho_1)$ and $\int_a^c f + \epsilon > \overline{R}(f, \rho_1)$.

$\int_c^b f - \epsilon < \underline{R}(f, \rho_2)$ and $\int_c^b f + \epsilon > \overline{R}(f, \rho_2)$.

Let $\rho = \rho_1 \cup \rho_2$.

$$\int_a^c f + \int_c^b f - 2\epsilon < \underline{R}(f, \rho_1) + \underline{R}(f, \rho_2) - \underline{R}(f, \rho) \leq \underline{\int_a^b} f. \quad (1)$$

$$\int_a^c f + \int_c^b f + 2\epsilon > \overline{R}(f, \rho_1) + \overline{R}(f, \rho_2) - \overline{R}(f, \rho) \geq \overline{\int_a^b} f. \quad (2)$$

From (1) and (2) we get

$$\int_a^c f + \int_c^b f - 2\epsilon < \underline{\int_a^b} f \leq \overline{\int_a^b} f < \int_a^c f + \int_c^b f + 2\epsilon .$$

$\because \epsilon$ is an arbitrary, then

$$\int_a^c f + \int_c^b f \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \int_a^c f + \int_c^b f.$$

$$\Longrightarrow \underline{\int_a^b} f = \overline{\int_a^b} f \Longrightarrow f \text{ is } R\text{-integrable on } [a, b].$$

Theorem 6.18: If f is a continuous function on $[a, b]$, then f is R-integrable on $[a, b]$.

Proof: $\because [a, b]$ is a compact set and $f: [a, b] \to R$ is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$ (if $f: X \to R$ is continuous and X is compact $\Longrightarrow f$ is uniformly continuous).

$\therefore \forall \epsilon > 0, \exists \delta > 0$ such that $\forall x, y \in [a, b]$ if $|x - y| < \delta$.

$\Longrightarrow |f(x) - f(y)| < \epsilon$

Let $\rho = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition on $[a, b]$ such that

$$\Delta x_i = \frac{b-a}{n} \quad \forall i = 1, \dots, n.$$

Let $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$

$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$

$\because f$ is conts on $[x_{i-1}, x_i]$

$\Longrightarrow \exists z_i, l_i \in [x_{i-1}, x_i]$ such that

$m_i = f(z_i), M_i = f(l_i), \quad \forall i = 1, \dots, n.$

$\because \delta > 0 \Longrightarrow \exists\, n \in N$ such that

$n\delta > (b-a) \Longrightarrow \frac{b-a}{n} < \delta.$

$\because \frac{b-a}{n} < \delta \Longrightarrow |z_i - l_i| < \delta, \quad \forall i = 1, \dots, n.$

$\because f$ is uniformly conts on $[a, b]$.

$\Longrightarrow |f(z_i) - f(l_i)| < \frac{\epsilon}{b-a}$

$\Longrightarrow |m_i - M_i| = |M_i - m_i| < \frac{\epsilon}{b-a}.$

$\therefore \overline{R}(f, \rho) - \underline{R}(f, \rho) < \epsilon \Longrightarrow f$ is R-integrable on $[a, b]$.

Remark 6,19: The converse of above theorem is not true. Consider the following example.

Example 6.20: Let $f: [0,2] \to R$ be a function such that

$$f(x) = \begin{cases} 2 & x \neq 1 \\ 1 & x = 1 \end{cases}$$

Then, f is R-integrable on $[0,2]$, but not continuous on $[0,2]$.

Solution:

Let $\rho = 0, \frac{2.1}{n}, \frac{2.2}{n}, \frac{2.3}{n}, \dots, \frac{2(n-1)}{n}, \frac{2n}{n} = 2.$

$\underline{R}(f, \rho) = \sum_{i=1}^{n} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$.

$= 2.\frac{2}{n} + 2.\frac{2}{n} + \cdots + 2.\frac{2}{n} = \frac{2}{n}$.

$= \frac{2}{n}\{2 + 2 + 2 + \cdots + (1)\} = \frac{2}{n}\{2(n-1) + 1\}.$

$= \frac{2}{n}\{2n - 1\} = 4 - \frac{2}{n}$.

$\underline{\int_0^2} f = \sup\, \{\underline{R}(f, \rho)\} = 4$

$\overline{R}(f,\rho) = \sum_{i=1}^{n} M_i \Delta x_i = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n \Delta x_n$

$= \sum_{i=1}^{n} 2.\frac{2}{n} = \frac{4}{n}.n = 4$

$\because \overline{\int_0^2} f = \underline{\int_0^2} f \Rightarrow \int_0^2 f = 4$

$\Rightarrow f$ is R-integrable on $[0,2]$.

But f is not conts on $[0,2]$ at $x = 1$,

And since $\langle \frac{1}{n} + 1 \rangle \to 1$ in $[0,2]$.

But $f\left(\frac{1}{n} + 1\right) = 2 \nrightarrow f(1) = 1$ in R.

6.4 Some Properties of *R*-Integrals (بعض خواص تكامل ريمان)

(1) If f is R-integrable on $[a,b]$, and

$f(x) \geq 0, \ \forall x \in [a,b]$, then $\int_a^b f \geq 0$.

Proof: $\because f(x) \geq 0 \ \forall x \in [a,b]$

$\Rightarrow \underline{R}(f,\rho) \geq 0$ for any partition on $[a,b]$.

$\underline{\int_a^b} f \geq 0$ since f is R-integrable

$\Rightarrow \underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f \geq 0 \Rightarrow \int_a^b f \geq 0.$

(2) If f_1 and f_2 are R-integrable on $[a,b]$ and $f_1 \leq f_2$, then $\int_a^b f_1 \leq \int_a^b f_2$.

Proof: Let $h = f_2 - f_1$.

$\because f_1(x) \leq f_2(x), \quad \forall x \in [a,b] \Rightarrow h(x) \geq 0 \quad \forall x \in [a,b]$.

$\Rightarrow \int_a^b h \geq 0 \Rightarrow \int_a^b (f_2 - f_1) \geq 0 \Rightarrow \int_a^b (f_2) + \int_a^b (-f_1) \geq 0.$

$\Rightarrow \int_a^b (f_2) - \int_a^b (f_1) \geq 0 \Rightarrow \int_a^b (f_2) \geq \int_a^b (f_1).$

$\Rightarrow \int_a^b (f_1) \leq \int_a^b (f_2).$

Or

$\because f_1(x) \leq f_2(x) \quad \forall x \in [a,b] \Rightarrow m_{1i} \leq m_{2i}$

$\Rightarrow \underline{R}(f_1,\rho) \leq \underline{R}(f_2,\rho) \Rightarrow \underline{\int_a^b}(f_1) \leq \underline{\int_a^b}(f_2)$ and since

f_1 and f_2 are R −integrable $\Rightarrow \int_a^b(f_1) \leq \int_a^b(f_2)$.

(3) Let f and h be a functions defined on $[\mathrm{a,b}]$ such that fh and h are R-integrable on $[\mathrm{a,b}]$ If $h \geq 0$ and m, M are constant such that $m \leq f \leq M$, then $m\int_a^b h \leq \int_a^b hf \leq M\int_a^b h$.

Proof: Check.

(4) If f is R-integrable on $[a,b]$, then $|f|$ is R-integrable on $[a,b]$ and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Proof: Let $m'_i = \inf\{|f| \,\big|\, x \in [x_{i-1},x_i]\}$ and $m_i = \inf\{f \,\big|\, x \in [x_{i-1},x_i]\}$ $M_i = \sup\{|f| \,\big|\, x \in [x_{i-1},x_i]\}$ and $M_i = \sup\{f \,\big|\, x \in [x_{i-1},x_i]\}$.

$\overline{R}(|f|,\rho) - \underline{R}(|f|,\rho) < \epsilon \Rightarrow |f|$ is R-integrable on $[a,b]$.

$\because -f \leq |f| \; and \; f \leq |f| \Rightarrow -\int_a^b f \leq \int_a^b |f|$ and

$\int_a^b f \leq \int_a^b |f| \Rightarrow \left|\int_a^b f\right| \leq \int_a^b |f|$.

Remark 6.21: The converse of (4) is not true.

i.e. $|f|$ is R-integrable on $[a,b]$, but f is not R-integrable on $[a,b]$

Example 6.22: Let $f(x) = \begin{cases} 1 & \text{if } x \in Q \text{ in } [0,1] \\ -1 & \text{if } x \notin Q \text{ in } [0,1] \end{cases}$

Let $\rho = \{0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{n-1}{n}, \frac{n}{n} = 1\}$ be a partition on $[0,1]$.

$$\underline{R}(f,\rho) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (-1)\frac{1}{n} = \frac{-n}{n} = -1$$

$$\overline{R}(f,\rho) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} (1)\frac{1}{n} = \frac{n}{n} = 1$$

$\underline{R}(f,\rho) \neq \overline{R}(f,\rho)$.

$\Rightarrow \underline{\int_0^1} f = 1$ and $\overline{\int_0^1} f = 1 \Rightarrow f$ is not R-integrable on $[0,1]$.

Also, $|f(x)| = 1 \quad \forall x \in [0,1]$,

$$\underline{R}(|f|,\rho) = \overline{R}(|f|,\rho) = 1$$

$$\Longrightarrow \underline{\int_0^1}|f| = \overline{\int_0^1}|f| = 1 \Longrightarrow |f| \text{ is R-integrable on } [0,1]$$

(5) If f is R-integrable and non negative on $[a,d]$ and if b,c are points such that $a < b < c < d$, then $\int_b^c f \leq \int_a^d f$. (check)

(6) If f is R-integrable on $[a,b]$, then f^2 is also R−integrable on $[a,b]$.

Proof: Case (1): $f \geq 0$

Let $m^2{}_i = \inf \{f^2(x) \,|\, x \in [x_{i-1}, x_i]\}$

$M_i^2 = \sup\{f^2(x) \,|\, x \in [x_{i-1}, x_i]\}$

$\overline{R}(f^2,\rho) - \underline{R}(f^2,\rho) = \sum_{i=1}^{n}(M^2{}_i - m^2{}_i)\Delta x_i = \sum_{i=1}^{n}(M_i - m_i)(M_i + m_i)\Delta x_i \leq \sum_{i=1}^{n}(M_i - m_i)2M\Delta x_i = 2M\sum_{i=1}^{n}(M_i - m_i)\Delta x_i = 2M[\sum_{i=1}^{n} M_i - \sum_{i=1}^{n} m_i]$

$$2M\left(\overline{R}(f,\rho) - \underline{R}(f,\rho)\right) < 2M\frac{\epsilon}{2M} = \epsilon$$

$\therefore f^2$ is R-integrable on $[a,b]$

Case (2): $f < 0$

$\because f < 0 \Longrightarrow |f| > 0 \Longrightarrow |f|$ is R-integrable on $[a,b]$

$\Longrightarrow |f|^2 = f^2 \Longrightarrow f^2$ is R-integrable on $[a,b]$

(7) If f and h are R-integrable on $[a,b]$, then fh is also R-integrable on $[a,b]$.

Proof: $\because f$ and h are R-integrable on $[a,b]$ then

$f + h$ is R-integrable on $[a,b]$.

$(f+h)^2$ and f^2 and h^2 are R-integrable on $[a,b]$

$\frac{1}{2}(f+h)^2 - \frac{1}{2} f^2 - \frac{1}{2}h^2 = fh$ is R-integrable on $[a,b]$

(8) If f is R-integrable on $[a,b] and\ 0 < m \leq f \leq M$ then $\frac{1}{f}$ is R-integrable on $[a,b]$.

Proof: $\because$ f is R-integrable on $[a,b] \Longrightarrow \forall\epsilon > 0, \exists$ a partition on $\rho\ s.t$

$\underline{R}(f,\rho) - \overline{R}(f,\rho) < \epsilon \Longrightarrow \sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \epsilon$

$m \leq f \leq M \Longrightarrow f$ is bounded $\Longrightarrow \frac{1}{M} \leq \frac{1}{f} \leq \frac{1}{m} \Longrightarrow \frac{1}{f}$ is bounded

$\sum_{i=1}^{n}\left(\frac{1}{m_i} - \frac{1}{M_i}\right)\Delta x_i = \sum_{i=1}^{n}(\frac{M_i - m_i}{m_i M_i})\Delta x_i \leq \sum_{i=1}^{n}\left(\frac{M_i - m_i}{m^2}\right)\Delta x_i =$
$\frac{1}{m^2}\sum_{i=1}^{n}\left(\frac{1}{m_i} - \frac{1}{M_i}\right)\Delta x_i < \frac{1}{m^2} = \epsilon$

$\frac{1}{f}$ is R-integrable on $[a,b]$.

(9) If f and g are R-integrable, then

$\left[\int_a^b fg\right]^2 \leq \left[\int_a^b f^2\right]\left[\int_a^b g^2\right]$ (Cauchy Schwarz inequality)

Proof: Take $At^2 + 2Bt + C > 0, \quad \forall t$

Let $B = \int_a^b fg$ and $A = \int_a^b f^2$ and $C = \int_a^b g^2$

$\left[\int_a^b fg\right]^2 - \left[\int_a^b f^2\right]\left[\int_a^b g^2\right] \leq 0$

$\left[\int_a^b fg\right]^2 \leq \left[\int_a^b f^2\right]\left[\int_a^b g^2\right]$

(10) $\left[\int_a^b (f+g)^2\right]^{\frac{1}{2}} \leq \left[\int_a^b f^2\right]^{\frac{1}{2}} + \left[\int_a^b g^2\right]^{\frac{1}{2}}$ (Minkowski inequality)

Proof: Since $\int_a^b (f+g)^2 = \int_a^b f^2 + 2\int_a^b fg + \int_a^b g^2$

$\leq \int_a^b f^2 + 2\left[\int_a^b f^2\right]^{\frac{1}{2}}\left[\int_a^b g^2\right]^{\frac{1}{2}} + \int_a^b g^2.$

$\Longrightarrow \left[\int_a^b (f+g)^2\right] \leq \left[\left[\int_a^b f^2\right]^{\frac{1}{2}} + \left[\int_a^b g^2\right]^{\frac{1}{2}}\right]^2$

$\left[\int_a^b (f+g)^2\right]^{\frac{1}{2}} \leq \left[\int_a^b f^2\right]^{\frac{1}{2}} + \left[\int_a^b g^2\right]^{\frac{1}{2}}$

6.5 Riemann Stieltjes Integral (تكامل ستجلس ريمان)

Definition 6.23: Let $f: [a, b] \to R$ be a bounded function and

Let $g: [a, b] \to R$ be not decreasing function and

Let $\rho = \{a = x_0, x_1, x_2, \dots, x_{u-1}, x_u, \dots, x_{n-1} x_n = b\}$ be a partition on $[a, b]$

$\overline{RS}(f, \rho, g) = \sum_{i=1}^{n} M_i [g(x_i) - g(x_{i-1})]$

$\underline{RS}(f, \rho, g) = \sum_{i=1}^{n} m_i [g(x_i) - g(x_{i-1})]$

Where $m_i = \inf\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$

$M_i = \sup\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$

$\because$ g is not decreasing $\Longrightarrow g(x_i) - g(x_{i-1}) \geq 0 \quad \forall i$

$\underline{RS}(f, \rho, g) \leq \overline{RS}(f, \rho, g)$

Let $\underline{\int_a^b} f dg = \sup\{\underline{RS}(f, \rho, g) \,|\, \rho$ is a partition on $[a, b]\}$

$\overline{\int_a^b} f dg = \inf\{\overline{RS}(f, \rho, g) \,|\, \rho$ is a partition on $[a, b]\}$

$\Longrightarrow \underline{\int_a^b} f dg \leq \overline{\int_a^b} f dg$

If $\underline{\int_a^b} f dg = \overline{\int_a^b} f dg \Longrightarrow f$ is R-integrable w.r.t. g and is denoted by $\int_a^b f dg$.

Remarks 6.24: If ρ^* is a refinement of ρ ,then

(1) $\underline{RS}(f, \rho, g) \leq \underline{RS}(f, \rho^*, g)$

$\overline{RS}(f, \rho^*, g) \leq \overline{RS}(f, \rho, g)$

(2) If ρ_1 and ρ_2 are a partition of $[a, b]$, then

$\underline{RS}(f, \rho_1, g) \leq \overline{RS}(f, \rho_2, g)$

(3) If $\rho = \{a, b\}$, then

$$\underline{RS}(f,\rho,g) = m[g(b) - g(a)]$$

$$\overline{RS}(f,\rho,g) = M[g(b) - g(a)].$$