## 1. The real numbers

**Definition 1.1:** Let $S \neq \phi$ be any set. Then a function $*: S \times S \to S$ is called a binary operation on $S$. We will write the element

$$* (a_1, a_2), \forall (a_1, a_2) \in S \times S$$

by follows $a_1 * a_2$ where:

(i)* is commutative on $S$, if $a * b = b * a, \ \forall a, b \in S$.

(ii) * is associative on $S$, if $(a * b) * c = a * (b * c), \ \forall a, b, c \in S$.

**Example 1.1:** Let $R$ be the set of real numbers, and $*$ defined on $R$ as follows:

$$a * b = a^3 + b^3, \forall a, b \in R.$$

Then, $*$ is commutative, but not associative (why?)

**Definition 1.2:** A field is a nonempty set $F$ with two operators "+" addition and " $\cdot$ " multiplication which satisfy the following "field axioms" $(A), (B)$ and $(C)$.

$(A)$ Axioms for addition:

$(A_1)$ $\forall x, y \in F \Rightarrow x + y \in F,$

$(A_2)$ $x + y = y + x, \forall x, y \in F,$

$(A_3)$ $(x + y) + z = x + (y + z), \forall x, y, z \in F,$

$(A_4)$ $\exists !\, 0 \in F \ s.t\ , 0 + x = x + 0 = x, \forall x \in F,$

$(A_5)$ $\forall x \in F \ \exists !\, (-x) \in F, s.t, \ x + (-x) = (-x) + x = 0.$

$(B)$ Axioms for multiplication:

$(B_1)$ $\forall x, y \in F \Rightarrow x . y \in F,$

$(B_2)$ $x.y = y.x, \forall x, y \in F$,

$(B_3)$ $(x.y).z = x.(y.z), \forall x, y, z \in F$,

$(B_4)$ $\exists!\, 1 \in F$ $s.t$ $1.x = x.1 = x, \forall x \in F$,

$(B_5)$ $\forall x \in F$ and $x \neq 0, \exists!\, (1/x) \in F$ $s.t$ $x.(1/x) = (1/x).x = 1$.

$(C)$ The distributive law:

$x.(y+z) = x.y + x.z, \forall x, y, z \in F$.

**Example 1.2:** $(R, +, .)$ is a field ($R$ real numbers)

$(Q, +, .)$ is a field ($Q$ rational numbers)

| $\mathbb{Q}_1$: Give example of not filed. |
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## *Order sets:*

**Definition 1.3:** Let $S$ be any set and $\leq, \subset, S^2$ be any relation defined on $S$. Then $\leq$ is partial order relation or order relation. If the following properties hold:

1. $\forall\, a \in S, a \leq a$ (reflexive).
2. $\forall\, a, b \in S$, if $a \leq b$ and $b \leq a \Rightarrow a = b$ (antisymmetric).
3. $\forall\, a, b, c \in S$, if $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitive).

**Definition 1.4:** Let $\varphi \neq S$ be any set and let $\leq$ be a relation on $S$, then we say that a pair $(S, \leq)$ is an ordered set (or partial ordered set) if $\leq$ is a partial order relation on $S$.

**Example 1.3:** Let $(Z, \leq), (P(A), \subseteq)$ are ordered sets. Where $P(A) = \{B : B \subset A\}$.

**Definition 1.5:** Let $(S, \leq)$ be an ordered set and $\forall a, b \in S$ either $a \leq b$ or $b \leq a$, then we say that the ordered pair $(S, \leq)$ is totally ordered set or ($a$ and $b$ are comparable).

**Example 1.4:**

1. $(Z, \leq)$ is totally ordered set
2. Let $A = \{1,2,3,4,5,6,7,8,9\}$ and $B = \{2,4,8\}$

   Let $\alpha$ be a relation on $A, B$ defined as follows:

   $\alpha = \{(a, b) \in A \times A\}$

Then $(A, \alpha)$ is not totally ordered set, because not every two elements in $A$ are comparable. $(B,\ A/B)$ is totally ordered set.

**Definition 1.5:** we say that $(S, \leq)$ is well-ordered set if $\forall \varphi \neq A \subset S, A$ has a smallest element in $A$. (i.e.), $\exists\ a \in A$ s.t $a \leq x, \forall x \in A$.

**Example 1.5**: $(N, \leq)$ is a well- ordered set.

- Give example of not well-ordered set?

## Bounded sets:

**Definition 1.7**: Let $(S, \leq)$ be an ordered set and $E \subset S$.

1. If $\exists\ a \in S$ s.t $x \leq a, \forall x \in E$, we say that $E$ is bounded above and call $a$ an upper bound of $E$.
2. An upper bound a of $E$ is called the least upper bound of $E$ (l.u.b ($E$)) or supremum of $E$ (sup(E)) if $a \leq y, \forall y$ upper bound of $E$.
3. If $\exists\ b \in S$ s.t $x \geq b, \forall x \in E$, we say that $E$ is bounded below and call $b$ a lower bound of $E$.
4. A lower bound $b$ of $E$ is called the greatest lower bound of $E$ ($g.L.b$ ($E$)) or infimum of $E$ (inf($E$)) if $b \geq y, \forall y$ lower bound of $E$.
5. $E$ is called bounded if $E$ is bounded above and bounded below.

**Example 1.6:**

1. Let $S = R$ and $E = (-\infty, 0)$, then $E$ is bounded above, since $\exists\ 0 \in R$ s.t $x \leq 0, \forall x \in E$

$\therefore$ $\mathrm{L.u.b(E)} = \sup\,(\mathrm{E}) = 0 \notin \mathrm{E}$

2. Let $S = Q$ and $\mathrm{E} = \{\dots, -3, -2, -1\}$. Then $E$ is bounded above, since $\exists -1(\text{or } 0,1,4, \dots) \in Q$ s.t $x \leq -1$ $(\text{or } 0,1,4, \dots) \forall x \in E$.
   $\therefore$ $\mathrm{L.u.b(E)} = \sup\,(\mathrm{E}) = -1 \in \mathrm{E}$
3. Let $S = R$ and $E = [-2, \infty)$, then $E$ is bounded below, since $\exists -2$ $(\text{or} -1, -3, \dots) \in S$ s.t $x \geq -2, \forall x \in E \Rightarrow g.\mathrm{L.b(E)} = \inf\,(\mathrm{E}) = -2 \in \mathrm{E}$
4. Let $S = Q$ and $\mathrm{E} = \{1,2,3,4, \dots\}$. Is E bounded below? (Example).
5. Let $S = R$ and $\mathrm{E} = (-1,5)$. Then $E$ is bounded above and $\sup(\mathrm{E}) = 5 \notin \mathrm{E}$. Also E is bounded below and $\inf(\mathrm{E}) = -1 \notin E$. Thus E is bounded.
6. Let $S = Q$ and $E_1 = \{x \in Q^+ | x^2 < 2\}$ and $E_2 = \{x \in Q^+ | x^2 > 2\} \Rightarrow E_1$ is bounded above. Since $\sqrt{2} \notin Q \Rightarrow E_1$ has no least upper bound (sup) in Q. Also, $E_2$ is bounded below, since $\sqrt{2} \notin Q \Rightarrow E_2$ has no greatest lower bound (inf) in Q.
7. Let $S = Q$ and $E = \left\{\frac{1}{n} | n = 1,2,3, \dots\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. Then $E$ is bounded. $\sup(\mathrm{E}) = 1 \in E$ and $\inf\,(\mathrm{E}) = 0 \notin \mathrm{E}$.

## *The completeness axiom*

Every bounded above set has the least upper bound.

*Equivalently:*

Every bounded below set has the greatest lower bound.

**Definition 1.8:** An ordered set $S$ is said to have the least upper bound property (or is said to be complete) if every non-empty bounded above subset $E$ of S has a supremum in $S$.

**Example 1.7:**

1. The real numbers $R$ is a complete order field.
2. Let $Q$ be an ordered set and $E = \{x \in Q^+ | x^2 < 2\}$, then $E \neq \varphi$ (since $1 \in \mathrm{E}$ ) and $E$ is bounded above

   $\because \sup(\mathrm{E}) = \sqrt{2} \notin Q \Rightarrow Q$ is not complete.