Chapter five

Numerical Differentiation and Integration

B. Numerical Differentiation of Newton Backward Formula

$$f(x)=f(x_n)+(x-x_n)\frac{\nabla f_n}{1!\,h}+(x-x_n)(x-x_{n-1})\frac{\nabla^2 f_n}{2!\,h^2}+\cdots$$
$$+(x-x_n)(x-x_{n-1})\ldots(x-x_1)\frac{\nabla^n f_n}{n!\,h^n}\ldots\ldots\ldots\ldots(6)$$

Let $\frac{x-x_n}{h}=q \Longrightarrow x-x_n=hq.$

$x-x_{n-1}=x-(x_n-h)=x-x_n+h=hq+h=h(q+1)$

$x-x_{n-2}=x-(x_n-2h)=x-x_n+2h=hq+2h=h(q+2)$

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In general

$$\boldsymbol{x-x_{n-k}=h(q+k)} \qquad ,\boldsymbol{k=0,1,2,3},\ldots$$

Substitute the last relation in to (1) we obtain :

$$f(x)=f(x_n)+q\nabla f_n+q(q+1)\frac{\nabla^2 f_n}{2!}+\cdots+q(q+1)\ldots(q+n-1)\frac{\nabla^n f_n}{n!}$$

$$f(x)=f(x_n)+q\nabla f_n+\frac{1}{2}(q^2+q)\nabla^2 f_n+\frac{1}{6}(q^3+3q^2+2q)\nabla^3 f_n+\cdots$$

$$f'(x)=\frac{df}{dq}.\frac{dq}{dx}=\frac{1}{h}.\frac{df}{dq}$$

$$f'(x) = \frac{1}{h}\Big[\nabla f_n + \frac{1}{2}(2q+1)\nabla^2 f_n + \frac{1}{6}(3q^2+6q+2)\nabla^3 f_n + \frac{1}{12}(2q^3+9q^2+11q+3)\nabla^4 f_n + \cdots\Big] \ldots (7)$$

$$f''(x) = \frac{d^2 f}{dq^2}.\frac{d^2 f}{dx^2} = \frac{1}{h^2}.\frac{d^2 f}{dq^2}$$

$$f''(x) = \frac{1}{h^2}\Big[\nabla^2 f_n + (q+1)\nabla^3 f_n + \frac{1}{12}(6q^2+18q+11)\nabla^4 f_n + \cdots\Big] \ldots (8)$$

If $x = x_i$

Set $x_i = x_n \Longrightarrow q = 0\,, f(x_i) = f(x_n)$

$$f'(x_n) = \frac{1}{h}\Big[\nabla f_n - \frac{1}{2}\nabla^2 f_n + \frac{1}{3}\nabla^3 f_n + \frac{1}{4}\nabla^4 f_n + \cdots\Big] \ldots (9)$$

$$f''(x_n) = \frac{1}{h^2}\Big[\nabla^2 f_0 - \nabla^3 f_0 + \frac{11}{12}\nabla^4 f_0 - \frac{5}{6}\nabla^5 f_0 \ldots\Big] \ldots (10)$$